

# Factorization of stochastic maps using the Stinespring representations

Carlo Pandiscia

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## Abstract

In this work, we investigate the existence of a factorization for a unital completely positive map, between non-commutative probability space which do not change the expectation values of the events. These maps are called in literature stochastic maps. Using the Stinespring representations of completely positive map and assuming the existence of anti-unitary operator on Hilbert space related to these representations which satisfying some modular relations, we prove that stochastic maps with adjoint, admits a factorization.

## 1 Introduction

We study the existence of a factorization for a preserving Markov operators, question posed and debated by Anantharaman-Delaroche in [2]. The factorization problem is closely connected with the existence of a reversible dilation of a quantum dynamical system in the sense of Kummerer [11], since Haagerup and Musat they proved in [9] that the factorization property is equivalent to the existence of a reversible dilation of quantum dynamical system.

The preserving Markov operator between commutative probability spaces are factorizable and each deterministic map *i.e.* multiplicative preserving Markov operator between generic probability space have this property since they admits a reversible dilation (see e.g. [9], [11], [16]).

In this paper, we give a constructive methods for to determine a factorization of preserving Markov map using the Stinespring representations and assuming the existence of anti-unitary operator  $\hat{J}$  on Hilbert space of the Stinespring representation. In commutative and deterministic case we have a natural choice for the anti-unitary operator  $\hat{J}$  which happens to be a conjugation.

We want to underline that the problem of to establish when a map admits a factorization and hence a reversible dilation, is a problem that still remains largely open, although it was posed many years ago [11].

this paper is organized as follows:

In section 2 we recall the main definitions and results on completely positive maps and quantum dynamical systems that can be found in [4], [15] and [17].

In section 3 Using the Stinespring representaions related to preserving Markov operator  $\Phi$  and assuming the existence of particular anti-unitary operator on Hilbert space of Stinespring representation, we prove that  $\Phi$  admits a factorization.

In section 4, After that we have introduced briefly the notion of the generalized conditional expectation of Accardi and Cecchini, which the reader can found in [1] and [5], we prove that if this map is a Umegaki conditional expectation [19], [6], then the preserving Markov operator admits a factorization.

## 2 Preliminaries

In this paper we consider the probability spaces  $(\mathfrak{M}, \varphi)$  constituted by a von Neumann algebra  $\mathfrak{M}$  and by its normal faithful state  $\varphi$ .

Let  $(\mathfrak{M}, \varphi)$  be a probability space, we set with  $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$  the GNS representation of the normal state  $\varphi$  and with  $(J_\varphi, \Delta_\varphi)$  the modular operators associated with the von Neumann algebra  $\pi_\varphi(\mathfrak{M})$  and with  $\{\sigma_t^\varphi\}$  its modular group.

Furthermore we set with  $\mathfrak{B}(\mathcal{H})$  the von Neumann algebra of bounded operators on Hilbert space  $\mathcal{H}$ .

A **stochastic map**  $\Phi : (\mathfrak{M}_1, \varphi_1) \rightarrow (\mathfrak{M}_2, \varphi_2)$  between probability space  $\{\mathfrak{M}_i, \varphi_i\}$  with  $i = 1, 2$ , is a normal unital completely positive map  $\Phi : \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$  with the following property  $\varphi_2 \circ \Phi = \varphi_1$ .

We have a normal unital completely positive map  $\Phi_\bullet : \pi_{\varphi_1}(\mathfrak{M}_1) \rightarrow \pi_{\varphi_2}(\mathfrak{M}_2)$  such that

$$\Phi_\bullet(\pi_{\varphi_1}(A)) = \pi_{\varphi_2}(\Phi(A))$$

for all  $A \in \mathfrak{M}_1$ .

Moreover, there is a linear contraction  $U_\Phi : \mathcal{H}_{\varphi_1} \rightarrow \mathcal{H}_{\varphi_2}$  defined as

$$U_\Phi \pi_{\varphi_1}(A) \Omega_{\varphi_1} = \pi_{\varphi_2}(\Phi(A)) \Omega_{\varphi_2} \quad (1)$$

for all  $A \in \mathfrak{M}_1$ .

Furthermore the Stochastic map  $\Phi$  admits a  $(\varphi_1, \varphi_2)$ -adjoint if there is a stochastic map  $\Phi^\sharp : (\mathfrak{M}_2, \varphi_2) \rightarrow (\mathfrak{M}_1, \varphi_1)$  such that

$$\varphi_2(B \Phi(A)) = \varphi_1(\Phi^\sharp(B) A)$$

for all  $A \in \mathfrak{M}_1$  and  $B \in \mathfrak{M}_2$ .

A stochastic map  $\Phi$  between two probability spaces is said to be a **deterministic map** whether it is a homomorphism of von Neumann algebras.

We have a fundamental proposition (see [1] and [15]):

**Proposition 1.** *Let  $\Phi : (\mathfrak{M}_1, \varphi_1) \rightarrow (\mathfrak{M}_2, \varphi_2)$  be a stochastic map, the following conditions are equivalent:*

- $\Phi$  admits  $(\varphi_1, \varphi_2)$ -adjoint
- $\Phi_\bullet \circ \sigma_t^{\varphi_1} = \sigma_t^{\varphi_2} \circ \Phi_\bullet \quad t \in \mathbb{R}$
- $J_{\varphi_2} U_\Phi = U_\Phi J_{\varphi_1}$ .

If the equivalent conditions of the previous proposition are satisfied, then we say that  $\Phi$  is a  $(\varphi_1, \varphi_2)$ -**preserving Markov map** [2].

We have the following definition:

**Definition 1.** *Let  $\Phi : (\mathfrak{M}_1, \varphi_1) \rightarrow (\mathfrak{M}_2, \varphi_2)$  be a  $(\varphi_1, \varphi_2)$ -preserving Markov map. We say that  $\Phi$  is a **factorizable map** if there exists a probability space  $\{\mathfrak{R}, \omega\}$  and two deterministic preserving Markov operators  $\alpha : (\mathfrak{M}_2, \varphi_2) \rightarrow (\mathfrak{R}, \omega)$  and  $\beta : (\mathfrak{M}_1, \varphi_1) \rightarrow (\mathfrak{R}, \omega)$  such that  $\Phi = \alpha^\sharp \circ \beta$ . The factorization  $(\alpha, \beta)$  is minimal if*

$$\mathfrak{R} = \alpha(\mathfrak{M}_2) \vee \beta(\mathfrak{M}_1)$$

where  $\alpha(\mathfrak{M}_2) \vee \beta(\mathfrak{M}_1)$  is the von Neumann algebra generated by  $\alpha(\mathfrak{M}_2)$  and  $\beta(\mathfrak{M}_1)$ .

We underline that a factorization  $(\alpha, \beta)$  of the preserving Markov operator  $\Phi$  determine a factorization of linear contraction  $U_\Phi$  since

$$U_\Phi = U_\alpha^* U_\beta$$

where  $U_\alpha$  and  $U_\beta$  are the linear isometries defined in (1) related to  $\alpha$  and  $\beta$  homomorphism.

In the following of the discussion, unless noted otherwise, we will consider the probability spaces  $(\mathfrak{M}, \Omega)$  in standard form *i.e.* concrete von Neumann algebra  $\mathfrak{M} \subset \mathfrak{B}(\mathcal{H})$  with  $\Omega$  cyclic and separating vector of Hilbert space  $\mathcal{H}$  and stochastic map  $\Phi : (\mathfrak{M}_1, \Omega_1) \rightarrow (\mathfrak{M}_2, \Omega_2)$  such that

$$\langle \Omega_2, \Phi(A) \Omega_2 \rangle = \langle \Omega_1, A \Omega_1 \rangle$$

for all  $A \in \mathfrak{M}_1$ .

For a preserving Markov operator  $\Phi : (\mathfrak{M}_1, \Omega_1) \rightarrow (\mathfrak{M}_2, \Omega_2)$ , its dual map [14]  $\Phi' : (\mathfrak{M}'_2, \Omega_2) \rightarrow (\mathfrak{M}'_1, \Omega_1)$  such that

$$\langle A\Omega_1 | \Phi'(Y)\Omega_1 \rangle = \langle \Phi(A)\Omega_2 | Y\Omega_2 \rangle$$

for all  $A \in \mathfrak{M}_1$  and  $Y \in \mathfrak{M}'_2$  is defined as

$$\Phi'(Y) = J_1 \Phi^\#(J_2 Y J_2) J_1$$

We observe that if  $(\mathfrak{M}, \varphi)$  is a probability space then  $\varphi$  is factorizable.

Indeed, we can define two preserving Markov maps  $\alpha, \beta : (\mathfrak{M}, \varphi) \rightarrow (\pi_\varphi(\mathfrak{M}) \overline{\otimes} \pi_\varphi(\mathfrak{M}), \Omega_\varphi \otimes \Omega_\varphi)$  where for any  $a \in \mathfrak{M}$ :

$$\alpha(a) = \pi_\varphi(a) \otimes 1 \quad \text{and} \quad \beta(a) = 1 \otimes \pi_\varphi(a)$$

and  $\pi_\varphi(\mathfrak{M}) \overline{\otimes} \pi_\varphi(\mathfrak{M})$  is the von Neumann algebra of  $\mathfrak{B}(\mathcal{H}_\varphi \otimes \mathcal{H}_\varphi)$  weakly closure of the \*-algebra generated by elements  $\sum_i^n A_i \otimes B_i$ , with  $A_i, B_i \in \pi_\varphi(\mathfrak{M})$ .

Furthermore for the adjoint maps we have:

$$\alpha^\#(A \otimes B) = \langle \Omega_\varphi, B\Omega_\varphi \rangle A \quad \text{and} \quad \beta^\#(A \otimes B) = \langle \Omega_\varphi, A\Omega_\varphi \rangle B$$

for all  $A, B \in \pi_\varphi(\mathfrak{M})$  and  $\varphi(a)I = \beta^\#(\alpha(a)) = \alpha^\#(\beta(a))$  for all  $a \in \mathfrak{M}$ .

We recall briefly the Stinespring representations associated to unital completely positive maps [17].

Let  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  be a concrete von Neumann algebra of  $\mathcal{B}(\mathcal{H}_1)$  and  $\mathcal{B}(\mathcal{H}_2)$  respectively.

Let  $\Phi : \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$  be a normal unital completely positive map

On the algebraic tensor  $\mathfrak{M}_1 \otimes \mathcal{H}_2$  we can define a semi-inner product by

$$\langle A \otimes h | X \otimes k \rangle = \langle h, \Phi(A^* X) k \rangle$$

for all  $A, X \in \mathfrak{M}_1$  and  $h, k \in \mathcal{H}_2$ .

Furthermore the Hilbert space  $\mathcal{L}_\Phi$  is the completion of the quotient space  $\mathfrak{M}_1 \overline{\otimes}_\Phi \mathcal{H}_2$  of  $\mathfrak{M}_1 \otimes \mathcal{H}_2$  by the linear subspace

$$\mathcal{V} = \{l \in \mathfrak{M}_1 \otimes \mathcal{H}_2 : \langle l | l \rangle = 0\} \quad (2)$$

with inner product induced by  $\langle \cdot, \cdot \rangle$ .

We shall denote the image at  $A \otimes h \in \mathfrak{M}_1 \otimes \mathcal{H}_2$  in  $\mathfrak{M}_1 \overline{\otimes}_\Phi \mathcal{H}_2$  by  $A \overline{\otimes}_\Phi h$ , so that we have

$$\langle A \overline{\otimes}_\Phi h, X \overline{\otimes}_\Phi k \rangle_{\mathcal{L}_\Phi} = \langle h, \Phi(A^* X) k \rangle$$

for all  $A, X \in \mathfrak{M}_1$  and  $h, k \in \mathcal{H}_2$ .

Moreover we can define a representation  $\sigma_\Phi : \mathfrak{M}_1 \rightarrow \mathcal{B}(\mathcal{L}_\Phi)$  defined by

$$\sigma_\Phi(A)(X \overline{\otimes}_\Phi h) = AX \overline{\otimes}_\Phi h,$$

for each  $A \overline{\otimes}_\Phi h \in \mathcal{L}_\Phi$  and  $V_\Phi h = \mathbf{1} \overline{\otimes}_\Phi h$  for each  $h \in \mathcal{H}_2$

Since  $\Phi$  is a unital map, the linear operator  $V_\Phi$  is an isometry with adjoint  $V_\Phi^*$  defined as

$$V_\Phi^* A \overline{\otimes}_\Phi h = \Phi(A)h$$

for all  $A \in \mathfrak{M}_1$  and  $h \in \mathcal{H}_2$ .

We can define the following linear operator  $\Lambda_\Phi : \mathcal{H}_1 \rightarrow \mathcal{L}_\Phi$ :

$$\Lambda_\Phi A \Omega_1 = A \overline{\otimes}_\Phi \Omega_2$$

for all  $A \in \mathfrak{M}_1$

We remark that

$$U_\Phi = V_\Phi^* \Lambda_\Phi$$

It is easy to prove that for each  $A \in \mathfrak{M}_1$  and  $h \in \mathcal{H}_2$  we have:

$$\Lambda_\Phi^* A \overline{\otimes}_\Phi h = A U_\Phi^* h$$

Furthermore

$$\Lambda_\Phi^* \sigma_\Phi(A) \Lambda_\Phi = A, \quad (3)$$

and

$$\Lambda_\Phi \Lambda_\Phi^* \in \sigma_\Phi(\mathfrak{M}_1)' \quad (4)$$

We have a new statement:

**Proposition 2.** *There is a normal representation  $\tau_\Phi : \mathfrak{M}'_2 \rightarrow \mathcal{B}(\mathcal{L}_\Phi)$  such that for each  $Y \in \mathfrak{M}'_2$  and  $A \overline{\otimes}_\Phi h \in \mathcal{L}_\Phi$*

$$\tau_\Phi(Y) A \overline{\otimes}_\Phi h = A \overline{\otimes}_\Phi Y h,$$

with

$$V_\Phi^* \tau_\Phi(Y) V_\Phi = Y \quad (5)$$

Furthermore

$$\tau_\Phi(\mathfrak{M}'_2) \subset \sigma_\Phi(\mathfrak{M}_1)', \quad (6)$$

and

$$V_\Phi V_\Phi^* \in \tau_\Phi(\mathfrak{M}'_2)' \quad (7)$$

*Proof.* We fix a vector  $l = \sum_i^n A_i \otimes h_i \in \mathfrak{M}_1 \otimes \mathcal{H}_2$  and we get the following linear functional on  $\mathfrak{M}'_2$

$$\omega_l(Y) = \langle l | Y l \rangle = \sum_{i,j}^n \langle h_i, \Phi(A_i^* A_j) Y h_j \rangle$$

for all  $Y \in \mathfrak{M}'_2$ .

The linear functional  $\omega_l$  is positive since

$$\omega_l(Y^* Y) = \sum_{i,j}^n \langle Y h_i, \Phi(A_i^* A_j) Y h_j \rangle \geq 0$$

then  $\omega_l$  is a continuous functional [3] with  $\omega_l(1) = \langle l | l \rangle$  and

$$|\omega_l(Y^* Y)| \leq \|Y\|^2 \omega_l(1) = \|Y\|^2 \langle l | l \rangle$$

For each  $Y \in \mathfrak{M}'_2$  we can define

$$\tau_o(Y) \sum_i^n A_i \otimes h_i = \sum_i^n A_i \otimes Y h_i.$$

We observe that if  $l = 0$  then  $\tau_o(Y)l = 0$  since  $\|\tau_o(Y)l\|^2 = |\omega_l(Y^* Y)| \leq \|Y\|^2 \langle l | l \rangle = 0$ .

Therefore  $\tau_o(\mathfrak{M}') \mathcal{V} \subset \mathcal{V}$  where  $\mathcal{V}$  is the linear space (2). It follows that  $\tau_\Phi : \mathfrak{M}'_2 \rightarrow \mathcal{B}(\mathcal{H})$  defined as

$$\tau_\Phi(Y) \sum_i^n A_i \overline{\otimes}_\Phi h_i = \sum_i^n A_i \overline{\otimes}_\Phi Y h_i,$$

for all  $Y \in \mathfrak{M}'_2$  is a well-defined representation of the von Neumann algebra  $\mathfrak{M}'_2$ .

Let  $\{Y_\alpha\}_\alpha$  be a net in  $\mathfrak{M}'$  such that  $Y_\alpha \rightarrow Y$  in  $\sigma$ -top, for each  $A \overline{\otimes}_\Phi h \in \mathcal{L}_\Phi$  we obtain:

$$\langle A \overline{\otimes}_\Phi h, \tau_\Phi(Y_\alpha) A \overline{\otimes}_\Phi h \rangle = \langle h, \Phi(A^* A) Y_\alpha h \rangle \rightarrow \langle h, \Phi(A^* A) Y h \rangle = \langle A \overline{\otimes}_\Phi h, \tau_\Phi(Y) A \overline{\otimes}_\Phi h \rangle$$

therefore the representation  $\tau_\Phi$  is  $\sigma$ -top continuous.

The others relationships are straightforward. □

If  $\Phi : (\mathfrak{M}_1, \Omega_1) \rightarrow (\mathfrak{M}_2, \Omega_2)$  is a preserving Markov operator then

$$\Lambda_\Phi^* \tau_\Phi(Y) \Lambda_\Phi = \Phi'(Y) \quad (8)$$

for all  $Y \in \mathfrak{M}'_2$ .

Indeed for each  $A \in \mathfrak{M}_1$  we have

$$\begin{aligned} \Lambda_\Phi^* \tau_\Phi(Y) \Lambda_\Phi A \Omega_1 &= \Lambda_\Phi^* \tau_\Phi(Y) A \overline{\otimes}_\Phi \Omega_2 = \Lambda_\Phi^* A \overline{\otimes}_\Phi Y \Omega_2 = A U_\Phi^* Y \Omega_2 \\ &= A J_1 U_\Phi^* J_2 Y J_2 \Omega_2 = A J_1 \Phi^\#(J_2 Y J_2) \Omega_1 = \\ &= J_1 \Phi^\#(J_2 Y J_2) J_1 A J_1 \Omega_1 = J_1 \Phi^\#(J_2 Y J_2) J_1 A \Omega_1 = \\ &= \Phi'(Y) A \Omega_1 \end{aligned}$$

We observe that  $\sigma_\Phi(\mathfrak{M}_1)$  and  $\tau_\Phi(\mathfrak{M}'_2)$  are von Neuman algebras, since the representations  $\sigma_\Phi : \mathfrak{M}_1 \rightarrow \mathcal{B}(\mathcal{L}_\Phi)$  and  $\tau_\Phi : \mathfrak{M}'_2 \rightarrow \mathcal{B}(\mathcal{L}_{\Phi'})$  are normal maps.

Furthermore, for each  $A \in \mathfrak{M}_2$  and  $Y \in \mathfrak{M}'_1$  we have:

$$V_\Phi^* \sigma_\Phi(A) \tau_\Phi(Y) V_\Phi = \Phi(A) Y \in \mathfrak{M}_2 \cdot \mathfrak{M}'_2 \subset \mathcal{B}(\mathcal{H}_2)$$

while

$$\Lambda_\Phi^* \sigma_\Phi(A) \tau_\Phi(Y) \Lambda_\Phi = A \Phi'(Y) \in \mathfrak{M}_1 \cdot \mathfrak{M}'_1 \subset \mathcal{B}(\mathcal{H}_1)$$

It follows that if  $\mathfrak{B} = \sigma(\mathfrak{M}_1) \vee \tau_\Phi(\mathfrak{M}'_2)$  is von Neumann algebra of  $\mathcal{B}(\mathcal{L}_\Phi)$  generated by  $\sigma(\mathfrak{M}_1)$  and  $\tau_\Phi(\mathfrak{M}'_2)$ , then we can define two unital completely positive maps  $E_i : \mathfrak{B} \rightarrow \mathfrak{M}_i \cdot \mathfrak{M}'_i$  with  $i = 1, 2$  such that for any  $T \in \mathfrak{B}$

$$E_1(T) = V_\Phi^* T V_\Phi \quad \text{and} \quad E_2(T) = \Lambda_\Phi^* T \Lambda_\Phi$$

with

$$\langle \Omega_i, E_i(T) \Omega_i \rangle = \langle \Omega_\Phi, T \Omega_\Phi \rangle$$

Furthermore, for any  $X_i \in \mathfrak{M}_2$  and  $h_i \in \mathcal{H}_1$  with  $i = 1, 2, \dots, n$ , we can define

$$W \sum_{i=1}^n X_i \overline{\otimes}_{\Phi^\#} h_i = \sum_{i=1}^n J_2 X_i J_2 \overline{\otimes}_{\Phi'} J_1 h_i \quad (9)$$

where

$$\begin{aligned} \|W \sum_{i=1}^n X_i \overline{\otimes}_{\Phi^\#} h_i\|^2 &= \sum_{i,j} \langle J_2 X_i J_2 \overline{\otimes}_{\Phi'} J_1 h_i, J_2 X_j J_2 \overline{\otimes}_{\Phi'} J_1 h_j \rangle = \\ &= \sum_{i,j} \langle J_1 h_i, \Phi'(J_2 X_i^* X_j J_2) J_1 h_j \rangle = \\ &= \sum_{i,j} \langle h_i, \Phi^\#(X_i^* X_j) h_j \rangle = \left\| \sum_{i=1}^n X_i \overline{\otimes}_{\Phi^\#} h_i \right\|^2 \end{aligned}$$

In other words, we have an anti-unitary operator  $W : \mathcal{L}_{\Phi^\#} \rightarrow \mathcal{L}_{\Phi'}$  such that

$$W^* \tau_{\Phi'}(A_1) W = \tau_{\Phi^\#}(J_1 A_1 J_1) \quad \text{and} \quad W^* \sigma_{\Phi'}(Y_2) W = \sigma_{\Phi^\#}(J_2 Y_2 J_2)$$

for all  $A_1 \in \mathfrak{M}_1$  and  $Y_2 \in \mathfrak{M}'_2$ .

### 3 Factorization of deterministic map

In this section we prove that each deterministic preserving Markov operator  $\Phi : (\mathfrak{M}_1, \Omega_1) \rightarrow (\mathfrak{M}_2, \Omega_2)$  is factorizable.

We set with  $(\mathcal{L}_{\Phi^\sharp}, \sigma_{\Phi^\sharp}, V_{\Phi^\sharp})$  and  $(\mathcal{L}_{\Phi^\sharp}, \tau_{\Phi^\sharp}, \Lambda_{\Phi^\sharp})$  the Stinespring representations of  $\Phi^\sharp$ , the adjoint of  $\Phi$ . We observe that the vector  $\Omega_{\Phi^\sharp} = 1 \otimes_{\Phi^\sharp} \Omega_1$  is separable for the von Neumann algebra  $\sigma_{\Phi^\sharp}(\mathfrak{M}_2)$  and we have a probability space  $(\sigma_{\Phi^\sharp}(\mathfrak{M}_2), \omega)$  where for any  $T \in \sigma_{\Phi^\sharp}(\mathfrak{M}_2)$  we have defined  $\omega(T) = \langle \Omega_{\Phi^\sharp}, T \Omega_{\Phi^\sharp} \rangle$ . Moreover the map  $\sigma_{\Phi^\sharp} : (\mathfrak{M}_2, \Omega_2) \rightarrow (\sigma_{\Phi^\sharp}(\mathfrak{M}_2), \omega)$  is a preserving Markov operator with adjoint  $\sigma_{\Phi^\sharp}^\sharp(T) = \Lambda_{\Phi^\sharp}^* T \Lambda_{\Phi^\sharp}$  for all  $T \in \sigma_{\Phi^\sharp}(\mathfrak{M}_2)$  since for each  $A, B \in \mathfrak{M}_2$  we have

$$\langle \Omega_{\Phi^\sharp}, \sigma_{\Phi^\sharp}(B) \sigma_{\Phi^\sharp}(A) \Omega_{\Phi^\sharp} \rangle = \langle \Omega_1, \Phi^\sharp(BA) \Omega_1 \rangle = \langle \Omega_2, BA \Omega_2 \rangle = \langle \Omega_2, \Lambda_{\Phi^\sharp}^* \sigma_{\Phi^\sharp}(B) \Lambda_{\Phi^\sharp} A \Omega_2 \rangle$$

The map  $\Theta : (\mathfrak{M}_1, \Omega_1) \rightarrow (\sigma_{\Phi^\sharp}(\mathfrak{M}_2), \omega)$  defined by

$$\Theta(A) = \sigma_{\Phi^\sharp}(\Phi(A))$$

for all  $A \in \mathfrak{M}_1$ , is a preserving Markov operator with adjoint  $\Theta^\sharp(T) = V_{\Phi^\sharp}^* T V_{\Phi^\sharp}$  for all  $T \in \sigma_{\Phi^\sharp}(\mathfrak{M}_2)$ . Indeed for each  $A \in \mathfrak{M}_1$  and  $B \in \mathfrak{M}_2$  we obtain:

$$\begin{aligned} \langle \Omega_{\Phi^\sharp}, \sigma_{\Phi^\sharp}(B) \Theta(A) \Omega_{\Phi^\sharp} \rangle &= \langle \Omega_{\Phi^\sharp}, \sigma_{\Phi^\sharp}(B \Phi(A)) \Omega_{\Phi^\sharp} \rangle = \langle \Omega_1, \Phi^\sharp(B \Phi(A)) \Omega_1 \rangle = \\ &= \langle \Omega_2, B \Phi(A) \Omega_2 \rangle = \langle \Omega_1, \Phi^\sharp(B) A \Omega_1 \rangle = \\ &= \langle \Omega_1, \Theta^\sharp(\sigma_{\Phi^\sharp}(B)) A \Omega_1 \rangle \end{aligned}$$

We have the following proposition:

**Proposition 3.** *Any deterministic preserving Markov operator  $\Phi : (\mathfrak{M}_1, \Omega_1) \rightarrow (\mathfrak{M}_2, \Omega_2)$  admits a factorization.*

*Proof.* We have that  $\Phi^\sharp(A) = \Theta^\sharp(\sigma_{\Phi^\sharp}(A))$  for all  $A \in \mathfrak{M}_1$  it follows that  $\Phi = \sigma_{\Phi^\sharp}^\sharp \circ \Theta$ .  $\square$

### 4 Factorization and Stinespring representations

We want to study the possibility of extending to any preserving Markov operators, the Stinespring representations methods used in the previous section to the deterministic case.

Let  $(\mathfrak{M}_1, \Omega_1)$  and  $(\mathfrak{M}_2, \Omega_2)$  be standard von Neumann algebras in Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$  respectively, and  $\Phi : (\mathfrak{M}_1, \Omega_1) \rightarrow (\mathfrak{M}_2, \Omega_2)$  a preserving Markov map with adjoint  $\Phi^\sharp : (\mathfrak{M}_2, \Omega_2) \rightarrow (\mathfrak{M}_1, \Omega_1)$ .

We consider the Stinespring representations  $(\mathcal{L}_{\Phi^\sharp}, \sigma_{\Phi^\sharp}, V_{\Phi^\sharp})$  and  $(\mathcal{L}_{\Phi^\sharp}, \tau_{\Phi^\sharp}, \Lambda_{\Phi^\sharp})$  of  $\Phi^\sharp$ .

We assume that there is anti-unitary operator  $\hat{J} : \mathcal{L}_{\Phi^\sharp} \rightarrow \mathcal{L}_{\Phi^\sharp}$  with the following property:

$$\hat{J} V_{\Phi^\sharp} = V_{\Phi^\sharp} J_1 \tag{10}$$

and we consider the von Neumann algebra  $\mathfrak{R}$  of  $\mathfrak{B}(\mathcal{L}_{\Phi^\sharp})$  generated by  $\sigma_{\Phi^\sharp}(\mathfrak{M}_2)$  and  $\hat{J}^* \tau_{\Phi^\sharp}(\mathfrak{M}_1') \hat{J}$ .

We remark that  $\Omega_{\Phi^\sharp} \in \mathcal{L}_{\Phi^\sharp}$  is cyclic vector for  $\mathfrak{R}$  since for each  $A_2 \in \mathfrak{M}_2$  and  $Y_1 \in \mathfrak{M}_1$  we obtain

$$\alpha(A_2) \hat{J}^* \tau_{\Phi^\sharp}(Y_1) \hat{J} \Omega_{\Phi^\sharp} = A_2 \overline{\otimes}_{\Phi^\sharp} J_1 Y_1 \Omega_1$$

We observe that

$$\mathfrak{M}_1 \subset V_{\Phi^\sharp}^* \mathfrak{R} V_{\Phi^\sharp} \quad \text{while} \quad \mathfrak{M}_2 \subset \Lambda_{\Phi^\sharp}^* \mathfrak{R} \Lambda_{\Phi^\sharp}$$

We have the following proposition:

**Proposition 4.** *Let  $\Phi : (\mathfrak{M}_1, \Omega_1) \rightarrow (\mathfrak{M}_2, \Omega_2)$  be a preserving Markov Operator and  $\hat{J} : \mathcal{L}_{\Phi^\sharp} \rightarrow \mathcal{L}_{\Phi^\sharp}$  which satisfies the relationships (10). If*

$$V_{\Phi^\sharp}^* \mathfrak{R} V_{\Phi^\sharp} \subset \mathfrak{M}_1 \quad \text{and} \quad \Lambda_{\Phi^\sharp}^* \mathfrak{R} \Lambda_{\Phi^\sharp} \subset \mathfrak{M}_2 \tag{11}$$

*then  $(\mathfrak{R}, \Omega_{\Phi^\sharp})$  is standard in  $\mathcal{L}_{\Phi^\sharp}$  and  $\Phi$  is factorizable.*

*Proof.* We prove that the vector  $\Omega_{\Phi^\#}$  is separating for  $\mathfrak{R}$ .

In fact, if  $R \in \mathfrak{R}$  with  $R\Omega_{\Phi^\#} = 0$ , then we can write that  $\Lambda_{\Phi^\#} R^* R \Lambda_{\Phi^\#} \Omega_2 = 0$  and from relationships (11) we have that  $\Lambda_{\Phi^\#} R^* R \Lambda_{\Phi^\#} \in \mathfrak{M}_2$  with  $(\mathfrak{M}_2, \Omega_2)$  standard in  $\mathcal{H}_2$ . It follows that  $\Lambda_{\Phi^\#} R^* R \Lambda_{\Phi^\#} = 0$  hence  $R\Lambda_{\Phi^\#} = 0$ .

In similar way we obtain that  $RV_{\Phi^\#} = 0$ .

For each  $A_2 \in \mathfrak{M}_2$  we have:

$$R\Lambda_{\Phi^\#} A_2 \Omega_2 = R\sigma_{\Phi^\#}(A_2)\Omega_{\Phi^\#} = 0$$

and repeating the reasoning for  $R\sigma_{\Phi^\#}(A_2) \in \mathfrak{R}$  we obtain:

$$R\sigma_{\Phi^\#}(A_2)\Lambda_{\Phi^\#} = 0 \quad \text{and} \quad R\sigma_{\Phi^\#}(A_2)V_{\Phi^\#} = 0$$

hence for each  $A_1 \in \mathfrak{M}_1$  result

$$R\sigma_{\Phi^\#}(A_2)V_{\Phi^\#}J_1A_1\Omega_1 = R\sigma_{\Phi^\#}(A_2)\beta(A_1)\Omega_{\Phi^\#} = 0$$

in other words  $RA_2\overline{\otimes}_{\Phi^\#}A_1\Omega_1 = 0$  for all  $A_2 \in \mathfrak{M}_2$  and  $A_1 \in \mathfrak{M}_1$ .

We consider  $\sigma_{\Phi^\#} : (\mathfrak{M}_2, \Omega_2) \rightarrow (\mathfrak{R}, \Omega_{\Phi^\#})$ , we have

$$\sigma_{\Phi^\#}^\#(T) = \Lambda_{\Phi^\#}^* T \Lambda_{\Phi^\#} \in \mathfrak{M}_2$$

for all  $T \in \mathfrak{R}$ , since

$$\langle \Omega_2, A_2 \Lambda_{\Phi^\#}^* T \Lambda_{\Phi^\#} \Omega_2 \rangle = \langle \Lambda_{\Phi^\#} A_2^* \Omega_2, T \Omega_{\Phi^\#} \rangle = \langle \Omega_{\Phi^\#}, \sigma_{\Phi^\#}(A_2) T \Omega_{\Phi^\#} \rangle$$

We can define an another stochastic map  $\beta : (\mathfrak{M}_1, \Omega_1) \rightarrow (\mathfrak{R}, \Omega_{\Phi^\#})$  as

$$\beta(A_1) = \widehat{J}^* \tau_{\Phi^\#}(J_1 A_1 J_1) \widehat{J}$$

for all  $A_1 \in \mathfrak{M}_1$ , with

$$\beta^\#(T) = V_{\Phi^\#}^* T V_{\Phi^\#} \in \mathfrak{M}_1$$

for all  $T \in \mathfrak{R}$ .

Indeed

$$\begin{aligned} \langle \Omega_{\Phi^\#}, \beta(A_1) T \Omega_{\Phi^\#} \rangle &= \langle \Omega_{\Phi^\#}, \widehat{J}^* \tau_{\Phi^\#}(J_1 A_1 J_1) \widehat{J} T \Omega_{\Phi^\#} \rangle = \langle \widehat{J}^* \tau_{\Phi^\#}(J_1 A_1^* J_1) \Omega_{\Phi^\#}, T \Omega_{\Phi^\#} \rangle = \\ &= \langle \widehat{J}^* V_{\Phi^\#} J_1 A_1^* J_1 \Omega_1, T \Omega_{\Phi^\#} \rangle = \langle V_{\Phi^\#} A_1^* \Omega_1, T \Omega_{\Phi^\#} \rangle = \\ &= \langle \Omega_1, A_1 V_{\Phi^\#}^* T \Omega_{\Phi^\#} \rangle = \langle \Omega_1, A_1 \beta^\#(T) \Omega_1 \rangle \end{aligned}$$

Furthermore, we have:

$$\beta^\#(\sigma_{\Phi^\#}(A_2)) = \Phi^\#(A_2)$$

for all  $A_2 \in \mathfrak{M}_2$ , hence  $\Phi = \sigma_{\Phi^\#}^\# \circ \beta$ . □

We observe that if  $\widehat{J}\Lambda_{\Phi^\#} = \Lambda_{\Phi^\#}J_2$  and  $\sigma_{\Phi^\#}(\mathfrak{M}_2) \subset \beta(\mathfrak{M}_1)'$  then the relationships (11) are satisfying, since  $\mathfrak{R}$  is generated by set of elements

$$\{\sigma_{\Phi^\#}(A_2) \cdot \beta(Y_1) : A_2 \in \mathfrak{M}_2 \quad Y_1 \in \mathfrak{M}_1\}$$

and by relationships (3), (4) and (8) we have:

$$\begin{aligned} \Lambda_{\Phi^\#}^* \sigma_{\Phi^\#}(A_2) \widehat{J}^* \tau_{\Phi^\#}(Y_1) \widehat{J} \Lambda_{\Phi^\#} &= \Lambda_{\Phi^\#}^* \sigma_{\Phi^\#}(A_2) \Lambda_{\Phi^\#} \Lambda_{\Phi^\#}^* \widehat{J}^* \tau_{\Phi^\#}(Y_1) \widehat{J} \Lambda_{\Phi^\#} = \\ &= \Lambda_{\Phi^\#}^* \sigma_{\Phi^\#}(A_2) \Lambda_{\Phi^\#} J_2 \Lambda_{\Phi^\#}^* \tau_{\Phi^\#}(Y_1) \Lambda_{\Phi^\#} J_2 = \\ &= A_2 \Phi(J_1 Y_1 J_1) \in \mathfrak{M}_2 \end{aligned}$$

for all  $A_2 \in \mathfrak{M}_2$  and  $Y_1 \in \mathfrak{M}'_1$ .

In similar way we have:

$$V_{\Phi^\#}^* \sigma_{\Phi^\#}(A_2) \hat{J}^* \tau_{\Phi^\#}(Y_1) \hat{J} V_{\Phi^\#} = \Phi^\#(A_2) J_1 Y_1 J_1 \in \mathfrak{M}_1$$

We see some applications of the previous proposition.

### Factorization in Abelian case

If  $\Phi : (\mathfrak{M}_1, \Omega_1) \rightarrow (\mathfrak{M}_2, \Omega_2)$  is a preserving Markov operator between commutative probability spaces then is factorizable.

Indeed, we consider the anti-unitary operator  $W : \mathcal{L}_{\Phi^\#} \rightarrow \mathcal{L}_{\Phi'}$  defined in (9) and the homomorphism  $\beta(A_1) = W^* \tau_{\Phi^\#}(J_1 A_1 J_1) W$  for all  $A_1 \in \mathfrak{M}_1$ .

In abelian case, because our von Neumann algebras are in standard form, we have that  $\mathfrak{M}_i = \mathfrak{M}'_i$  for  $i = 1, 2$  and  $\Phi^\# = \Phi'$  with

$$W^* \tau_{\Phi^\#}(\mathfrak{M}'_1) W = \tau_{\Phi^\#}(\mathfrak{M}'_1) = \tau_{\Phi^\#}(\mathfrak{M}_1)$$

since  $\tau_{\Phi^\#} = \tau_{\Phi'}$ .

We remark that the anti-unitary  $W$  is an involution *i.e.*  $W^2 = 1$ .

Hence, the von Neumann algebra  $\mathfrak{R}$  is generated by algebra  $\sigma_{\Phi^\#}(\mathfrak{M}_2)$  and  $\tau_{\Phi^\#}(\mathfrak{M}'_1)$ . From relationship (6) and of the previous remark, we have that  $\Omega_{\Phi^\#}$  is a cyclic and separable vector for  $\mathfrak{R}$  and the pair  $(\sigma_{\Phi^\#}, \beta)$  is a minimal factorization of  $\Phi$ .

### Deterministic case

We consider again the deterministic case, we proof that there is a anti-unitary operator  $\hat{J}$  which satisfies the relationship (10) and

$$\hat{J}^* \tau_{\Phi^\#}(\mathfrak{M}'_1) \hat{J} \subset \sigma_{\Phi^\#}(\mathfrak{M}_2)$$

in other words that  $\mathfrak{R} = \sigma_{\Phi^\#}(\mathfrak{M}_2)$ .

Because  $\Omega_{\Phi^\#}$  is a cyclic vector for  $\sigma_{\Phi^\#}(\mathfrak{M}_2)'$  we can consider the following anti-linear map

$$\hat{J} T' \Omega_{\Phi^\#} := J_2 \Lambda_{\Phi^\#}^* T' \Lambda_{\Phi^\#} J_2 \overline{\otimes}_{\Phi^\#} \Omega_1 \quad T' \in \sigma_{\Phi^\#}(\mathfrak{M}_2)' \quad (12)$$

We remark that

$$\Lambda_{\Phi^\#}^* \sigma_{\Phi}(\mathfrak{M}_2)' \Lambda_{\Phi^\#} \subset (\Lambda_{\Phi^\#}^* \sigma_{\Phi}(\mathfrak{M}_2) \Lambda_{\Phi^\#})' = \mathfrak{M}'_2$$

because  $\Lambda_{\Phi^\#} \Lambda_{\Phi^\#}^* \in \sigma_{\Phi^\#}(\mathfrak{M}_2)'$ .

Furthermore, we have for any  $A_2 \in \mathfrak{M}_2$  and  $h_1 \in \mathcal{H}_1$  that

$$\hat{J}^* A_2 \overline{\otimes}_{\Phi^\#} h_1 = \Lambda_{\Phi^\#} J_2 A_2 U_{\Phi} h_1$$

since for any  $T' \in \sigma_{\Phi^\#}(\mathfrak{M}_2)'$  and  $A_2 \in \mathfrak{M}_2$ ,  $h_1 \in \mathcal{H}_1$  we have

$$\begin{aligned} \langle \hat{J}^* A_2 \overline{\otimes}_{\Phi^\#} h_1, T' \Omega_{\Phi^\#} \rangle &= \langle \hat{J} T' \Omega_{\Phi^\#}, A_2 \overline{\otimes}_{\Phi^\#} h_1 \rangle = \langle J_2 \Lambda_{\Phi^\#}^* T' \Lambda_{\Phi^\#} J_2 \overline{\otimes}_{\Phi^\#} \Omega_1, A_2 \overline{\otimes}_{\Phi^\#} h_1 \rangle = \\ &= \langle \Omega_1, \Phi^\#(J_2 \Lambda_{\Phi^\#}^* T'^* \Lambda_{\Phi^\#} J_2 A_2) h_1 \rangle = \langle U_{\Phi}^* A_2^* J_2 \Lambda_{\Phi^\#}^* T' \Omega_{\Phi^\#}, h_1 \rangle = \\ &= \langle \Lambda_{\Phi^\#} J_2 A_2 U_{\Phi} h_1, T' \Omega_{\Phi^\#} \rangle \end{aligned}$$

Moreover  $\hat{J}^* \hat{J} = \Lambda_{\Phi^\#} \Lambda_{\Phi^\#}^*$  and since  $\Phi$  is a multiplicative map we have  $\Lambda_{\Phi^\#} \Lambda_{\Phi^\#}^* = I$ .

Since  $\Phi$  is a multiplicative map we have  $U_{\Phi}^* A_2 U_{\Phi} = \Phi^\#(A_2)$  for all  $A_2 \in \mathfrak{M}_2$  and

$$\begin{aligned} \langle \hat{J}^* A_2 \overline{\otimes}_{\Phi^\#} h_1, \hat{J}^* B_2 \overline{\otimes}_{\Phi^\#} k_1 \rangle &= \langle \Lambda_{\Phi^\#} J_2 A_2 U_{\Phi} h_1, \Lambda_{\Phi^\#} J_2 B_2 U_{\Phi} k_1 \rangle = \langle B_2 U_{\Phi} k_1, A_2 U_{\Phi} h_1 \rangle = \\ &= \langle k_1, U_{\Phi}^* B_2^* A_2 U_{\Phi} h_1 \rangle = \langle k_1, \Phi^\#(B_2^* A_2) h_1 \rangle = \\ &= \langle B_2 \overline{\otimes}_{\Phi^\#} K_1, A_2 \overline{\otimes}_{\Phi^\#} h_1 \rangle \end{aligned}$$



It follows that  $\widehat{J}\widehat{J}^* = I$ .

We observe that for any  $R', T' \in \sigma_{\Phi^\#}(\mathfrak{M}_2)'$  and  $A_1 \in \mathfrak{M}_1$  we have

$$\begin{aligned}
\langle R'\Omega_{\Phi^\#}, \widehat{J}^*\tau_{\Phi^\#}(J_1 A_1 J_1)\widehat{J}T'\Omega_{\Phi^\#} \rangle &= \langle \tau_{\Phi^\#}(J_1 A_1 J_1)\widehat{J}^*T'\Omega_{\Phi^\#}, \widehat{J}R'\Omega_{\Phi^\#} \rangle = \\
&= \langle J_2\Lambda_{\Phi^\#}^*T'\Lambda_{\Phi^\#}J_2\overline{\otimes}_{\Phi^\#}J_1 A_1\Omega_1, J_2\Lambda_{\Phi^\#}^*R'\Lambda_{\Phi^\#}J_2\overline{\otimes}_{\Phi^\#}\Omega_1 \rangle = \\
&= \langle J_1 A_1\Omega_1, \Phi^\#(J_2\Lambda_{\Phi^\#}^*T'^*\Lambda_{\Phi^\#}\Lambda_{\Phi^\#}^*R'\Lambda_{\Phi^\#}J_2)\Omega_1 \rangle = \\
&= \langle J_1 A_1\Omega_1, U_{\Phi}^*J_2\Lambda_{\Phi^\#}^*T'^*\Lambda_{\Phi^\#}\Lambda_{\Phi^\#}^*R'\Omega_{\Phi^\#} \rangle = \\
&= \langle J_2U_{\Phi}A_1\Omega_1, J_2\Lambda_{\Phi^\#}^*T'^*\Lambda_{\Phi^\#}\Lambda_{\Phi^\#}^*R'\Omega_{\Phi^\#} \rangle = \\
&= \langle \Lambda_{\Phi^\#}^*T'^*\Lambda_{\Phi^\#}\Lambda_{\Phi^\#}^*R'\Omega_{\Phi^\#}, \Phi(A_1)\Omega_2 \rangle = \\
&= \langle \Lambda_{\Phi^\#}\Lambda_{\Phi^\#}^*R'\Omega_{\Phi^\#}, T'\sigma_{\Phi^\#}(\Phi(A_1)\Omega_{\Phi^\#}) \rangle = \\
&= \langle R'\Omega_{\Phi^\#}, \Lambda_{\Phi^\#}\Lambda_{\Phi^\#}^*\sigma_{\Phi^\#}(\Phi(A_1)T'\Omega_{\Phi^\#}) \rangle
\end{aligned}$$

Because  $\Omega_{\Phi^\#}$  is cyclic for  $\sigma_{\Phi^\#}(\mathfrak{M}_2)'$  we can write

$$\widehat{J}^*\tau_{\Phi^\#}(J_1 A_1 J_1)\widehat{J} = \Lambda_{\Phi^\#}\Lambda_{\Phi^\#}^*\sigma_{\Phi^\#}(\Phi(A_1)) = \sigma_{\Phi^\#}(\Phi(A_1))$$

thus

$$\widehat{J}^*\tau_{\Phi^\#}(\mathfrak{M}_1')\widehat{J} \subset \sigma_{\Phi^\#}(\mathfrak{M}_2)$$

Moreover for any  $A_2\overline{\otimes}_{\Phi^\#}h_1 \in \mathcal{L}_{\Phi^\#}$  we have

$$\Lambda_{\Phi^\#}^*\widehat{J}^*A_2\overline{\otimes}_{\Phi^\#}h_1 = J_2A_2U_{\Phi}h_1 = J_2\Lambda_{\Phi^\#}^*A_2\overline{\otimes}_{\Phi^\#}h_1$$

and

$$\begin{aligned}
V_{\Phi^\#}^*\widehat{J}^*A_2\overline{\otimes}_{\Phi^\#}h_1 &= V_{\Phi^\#}^*\Lambda_{\Phi^\#}J_2A_2U_{\Phi}h_1 = J_1\Phi^\#(A_2)h_1 = \\
&= J_1V_{\Phi^\#}^*A_2\overline{\otimes}_{\Phi^\#}h_1
\end{aligned}$$

We observe that the anti-unitary operator  $\widehat{J}$  is an involution since:

$$\begin{aligned}
\langle \widehat{J}^*A_2\overline{\otimes}_{\Phi^\#}h_1, \widehat{J}A_2\overline{\otimes}_{\Phi^\#}h_1 \rangle &= \langle \Lambda J_2A_2U_{\Phi}h_1, \widehat{J}A_2\overline{\otimes}_{\Phi^\#}h_1 \rangle = \\
&= \langle \Lambda J_2A_2U_{\Phi}h_1, \widehat{J}\Lambda A_2U_{\Phi}h_1 \rangle = \\
&= \langle \Lambda J_2A_2U_{\Phi}h_1, \Lambda J_2A_2U_{\Phi}h_1 \rangle
\end{aligned}$$

since  $A_2\overline{\otimes}_{\Phi^\#}h_1 = \Lambda_{\Phi^\#}\Lambda_{\Phi^\#}^*A_2\overline{\otimes}_{\Phi^\#}h_1 = \Lambda_{\Phi^\#}A_2U_{\Phi}h_1$ .

## 5 Factorization and generalized conditional expectation

We recall briefly the notion of generalized conditional expectation of Accardi and Cecchini [1].

Let  $(\mathfrak{M}, \varphi)$  be a probability space and  $\mathfrak{A}$  a von Neumann algebra with  $i : \mathfrak{M} \rightarrow \mathfrak{A}$  an injective homomorphism between von Neumann algebras.

We set with the space of normal  $\mathfrak{S}_\varphi$  the set:

$$\mathfrak{S}_\varphi = \{\omega \in \mathfrak{A}_* : \omega(i(a)) = \varphi(a) \text{ for all } a \in \mathfrak{M}\}$$

where  $\mathfrak{A}_*$  the predual of  $\mathfrak{A}$ .

Let  $(\mathcal{H}_s, \pi_s, J_s, \mathcal{P}_s)$  be a standard representation of algebra of von Neumann  $\mathfrak{A}$  [8], it is widely know that there is a unique  $\xi_\omega \in \mathcal{P}_s$  such that

$$\omega(X) = \langle \xi_\omega \pi_s(X) \xi_\omega \rangle \text{ for all } X \in \mathfrak{A}$$

We define the following isometry  $\nabla_\omega : \mathcal{H}_\varphi \rightarrow \mathcal{H}_s$ :

$$\nabla_\omega \pi_\varphi(a) \Omega_\varphi = \pi_s(i(a)) \xi_\omega \quad \text{for all } a \in \mathfrak{M}$$

and we obtain (see [1]) a unital completely positive map  $\mathcal{E}_\omega : \mathfrak{R} \rightarrow \mathfrak{M}$  such that

$$\pi_\varphi(\mathcal{E}_\omega(X)) = J_\varphi \nabla_\omega^* \pi_s(X) J_s \nabla_\omega J_\varphi \quad \text{for all } X \in \mathfrak{R}$$

Furthermore  $\mathcal{E}_\omega(i(a)) = a$  for all  $a \in \mathfrak{M}$  if, and only if  $J_s \nabla_\omega = \nabla_\omega J_\varphi$ .

We consider again the preserving Markov Operator  $\Phi : (\mathfrak{M}_1, \Omega_1) \rightarrow (\mathfrak{M}_2, \Omega_2)$  and the Stinespring representations  $(\mathcal{L}_{\Phi^\#}, \sigma_{\Phi^\#}, V_{\Phi^\#})$  and  $(\mathcal{L}_{\Phi^\#}, \tau_{\Phi^\#}, \Lambda_{\Phi^\#})$  related to adjoint map  $\Phi^\#$ .

Let  $\hat{J} : \mathcal{L}_{\Phi^\#} \rightarrow \mathcal{L}_{\Phi^\#}$  be an anti-unitary operator with the property (10) and we set with  $\mathfrak{R}$  the von Neumann algebra generated by  $\sigma_{\Phi^\#}(\mathfrak{M}_2)$  and  $\hat{J} \tau_{\Phi^\#}(\mathfrak{M}'_1) \hat{J}$ .

Moreover, let  $(\pi_s, \mathcal{H}_s, J_s, \mathcal{P}_s)$  be the standard representation of  $\mathfrak{R}$ , we define the following isometries  $\nabla_i : \mathcal{H}_i \rightarrow \mathcal{L}_{\Phi^\#}$  as

$$\begin{aligned} \nabla_1 A_1 \Omega_1 &= \pi_s(\beta(A_1)) \Omega_s & A_1 &\in \mathfrak{M}_1 \\ \nabla_2 A_2 \Omega_2 &= \pi_s(\sigma_{\Phi^\#}(A_2)) \Omega_s & A_2 &\in \mathfrak{M}_2 \end{aligned}$$

where  $\beta(A_1) = \hat{J}^* \tau_{\Phi^\#}(J_1 A_1 J_1) \hat{J}$  for all  $A_1 \in \mathfrak{M}_1$ .

We have a generalized conditional expectations  $\mathcal{E}_i : \mathfrak{R} \rightarrow \mathfrak{M}_i$  with  $i = 1, 2$  such that for each  $R \in \mathfrak{R}$

$$\mathcal{E}_i(R) = J_i \nabla_i^* J_s \pi_s(R) J_s \nabla_i J_i \quad (13)$$

Furthermore we have  $\mathcal{E}_1(\beta(A_1)) = A_1$  and  $\mathcal{E}_1(\sigma_{\Phi^\#}(A_2)) = A_2$  for all  $A_i \in \mathfrak{M}_i$  with  $i = 1, 2$  if, and only if

$$J_s \nabla_i = \nabla_i J_i \quad (14)$$

The vector  $\Omega_{\Phi^\#}$  is cyclic for  $\mathfrak{R}$  and we have the following:

**Proposition 5.** *If the relationships (14) hold then  $\Omega_{\Phi^\#}$  is a separating vector for  $\mathfrak{R}$ . Furthermore we have*

$$\langle \Omega_{\Phi^\#}, R \sigma_{\Phi^\#}(A_2) \rangle = \langle \Omega_2, \mathcal{E}_2(R) A_2 \Omega_2 \rangle \quad R \in \mathfrak{R} \quad A_2 \in \mathfrak{M}_2 \quad (15)$$

and

$$\langle \Omega_{\Phi^\#}, R \beta(A_1) \rangle = \langle \Omega_1, \mathcal{E}_1(R) A_1 \Omega_1 \rangle \quad R \in \mathfrak{R} \quad A_1 \in \mathfrak{M}_1 \quad (16)$$

in other words  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are adjoints maps of  $\beta$  and  $\alpha$  respectively.

*Proof.* The proof of separating property is similar the previous proposition. Indeed, let  $R$  belongs to  $\mathfrak{R}$  such that  $R \Omega_{\Phi^\#} = 0$ . From (13) we have that  $\nabla_i^* R^* R \nabla_i \Omega_i = 0$  for all  $i = 1, 2$ .

It follows that  $R \nabla_i = 0$  for all  $i = 1, 2$ .

For each  $A_2 \in \mathfrak{M}_2$  we obtain:

$$R \nabla_2 A_2 \Omega_2 = R \sigma_{\Phi^\#}(A_2) \Omega_{\Phi^\#} = 0$$

and repeating the argument for  $R \sigma_{\Phi^\#}(A_2) \in \mathfrak{R}$  we obtain:

$$R \sigma_{\Phi^\#}(A_2) \nabla_i = 0$$

and for each  $A_1 \in \mathfrak{M}_1$  we have

$$R \sigma_{\Phi^\#}(A_2) \nabla_1 A_1 \Omega_1 = R \sigma_{\Phi^\#}(A_2) \beta(A_1) \Omega_{\Phi^\#} = 0$$

hence  $R A_2 \overline{\otimes}_{\Phi^\#} A_1 \Omega_1 = 0$  for all  $A_i \in \mathfrak{M}_i$  with  $i = 1, 2$ .

We have for all  $R \in \mathfrak{R}$  and  $A_i \in \mathfrak{M}_i$   $i = 1, 2$

$$\langle \Omega_2, \mathcal{E}_2(R) A_2 \Omega_2 \rangle = \langle \Omega_2, \nabla_2^* R \nabla_2 A_2 \Omega_2 \rangle = \langle \Omega_s, \pi_s(R \sigma_{\Phi^\#}(A_2) \Omega_s) \rangle = \langle \Omega_{\Phi^\#}, R \sigma_{\Phi^\#}(A_2) \Omega_{\Phi^\#} \rangle$$

while

$$\langle \Omega_1, \mathcal{E}_1(R)A_1\Omega_1 \rangle = \langle \Omega_1, \nabla_1^* R \nabla_1 A_1\Omega_1 \rangle = \langle \Omega_s, \pi_s(R\beta(A_1)\Omega_s) \rangle = \langle \Omega_{\Phi^\#}, R\beta(A_1)\Omega_{\Phi^\#} \rangle$$

□

We can give the following proposition:

**Corollary 1.** *If the relationships (14) hold then  $\Phi$  is a factorizable map.*

*Proof.* We have that  $\Phi^\#(A_1) = \beta^\#(\alpha(A_1))$  for all  $A_1 \in \mathfrak{M}_1$ .

Indeed

$$\begin{aligned} \langle A_1\Omega_1, \beta^\#(\alpha(A_2))\Omega_1 \rangle &= \langle A_1\Omega_1, \nabla_1^* \pi_s(\sigma_{\Phi^\#}(A_2))\nabla_1\Omega_1 \rangle = \langle \pi_s(\beta(A_1))\Omega_s, \pi_s(\sigma_{\Phi^\#}(A_2))\Omega_s \rangle = \\ &= \langle \beta(A_1)\Omega_{\Phi^\#}, \sigma_{\Phi^\#}(A_2)\Omega_{\Phi^\#} \rangle = \langle 1 \otimes_{\Phi^\#} A_1\Omega_1, A_2 \otimes_{\Phi^\#} \Omega_1 \rangle = \\ &= \langle A_1\Omega_1, \Phi^\#(A_2)\Omega_1 \rangle \end{aligned}$$

□

We have an isometry  $\Xi : \mathcal{L}_{\Phi^\#} \rightarrow \mathcal{H}_s$  such that

$$\Xi A_2 \otimes_{\Phi^\#} A_1\Omega_1 = \pi_s(\sigma_{\Phi^\#}(A_2)\beta(A_1))\xi_\omega$$

for all  $A_i \in \mathfrak{M}_i$  with  $i = 1, 2$ , where

$$\langle \Omega_{\Phi^\#}, X\Omega_{\Phi^\#} \rangle = \langle \xi_\omega, \pi_s(X)\xi_\omega \rangle \quad \text{for all } X \in \mathfrak{R}$$

Moreover

$$\Xi V_{\Phi^\#} = \nabla_1 \quad \text{and} \quad \Xi \Lambda_{\Phi^\#} = \nabla_2$$

and if the (14) is hold, we can write a relationship between the anti-unitary  $\widehat{J}$  and the modular coniugation  $J_s$ :

$$\widehat{J}V_{\Phi^\#} = \Xi^* J_s \Xi V_{\Phi^\#}$$

We observe that for each  $R \in \mathfrak{R}$  we have:

$$V_{\Phi^\#}^* R V_{\Phi^\#} = \nabla_1^* \pi_s(R) \nabla_1 \quad \text{and} \quad \Lambda_{\Phi^\#}^* R \Lambda_{\Phi^\#} = \nabla_2^* \pi_s(R) \nabla_2$$

Indeed for each  $A_i, B_i \in \mathfrak{M}_i$  with  $i = 1, 2$  we can write:

$$\begin{aligned} \langle B_1\Omega_1, \nabla_1^* \pi_s(R) \nabla_1 A_1\Omega_1 \rangle &= \langle \xi_\omega, \pi_s(\beta(B_1^*))\pi_s(R)\pi_s(\beta(A_1))\xi_\omega \rangle = \\ &= \langle \Omega_{\Phi^\#}, \beta(B_1^*)R\beta(A_1)\Omega_{\Phi^\#} \rangle = \\ &= \langle \widehat{J}V_{\Phi^\#}J_1B_1\Omega_1, R\widehat{J}V_{\Phi^\#}J_1A_1\Omega_1 \rangle = \\ &= \langle V_{\Phi^\#}B_1\Omega_1, R V_{\Phi^\#}A_1\Omega_1 \rangle = \\ &= \langle B_1\Omega_1, V_{\Phi^\#}^* R V_{\Phi^\#}A_1\Omega_1 \rangle \end{aligned}$$

while

$$\begin{aligned} \langle B_2\Omega_2, \nabla_2^* \pi_s(R) \nabla_2 A_2\Omega_2 \rangle &= \langle \xi_\omega, \pi_s(\sigma_{\Phi^\#}(B_2^*))\pi_s(R)\pi_s(\sigma_{\Phi^\#}(A_2))\xi_\omega \rangle = \\ &= \langle \Lambda_{\Phi^\#}B_2\Omega_2, R\Lambda_{\Phi^\#}A_2\Omega_2 \rangle = \\ &= \langle B_2\Omega_2, \Lambda_{\Phi^\#}^* R \Lambda_{\Phi^\#}A_2\Omega_2 \rangle \end{aligned}$$

We can give a simple remark:

**Remark 1.** *We have that  $J_s \nabla_i = \nabla_i J_i$  with  $i = 1, 2$  if, and only if*

$$\mathcal{E}_1(R) = V^* R V \quad \text{and} \quad \mathcal{E}_2(R) = \Lambda^* R \Lambda$$

for all  $R \in \mathfrak{R}$ .

## 6 Conclusion

In this paper we have given a simple method to determine when a preserving Markov operator is factorizable. It is based on the appropriate selection of an anti-unitary operator  $\hat{J}$  on Hilbert space of Stinespring representation  $\mathcal{L}_{\Phi\#}$ .

A useful tool to establish the anti-unitary operator  $\hat{J}$  be found in [12] and [13] since there is a strong connection between conjugation operator and antilinear Jordan map of von Neumann algebra. Indeed, for each antilinear Jordan map  $\gamma : \sigma_{\Phi\#}(\mathfrak{M}_2)' \rightarrow \sigma_{\Phi\#}(\mathfrak{M}_2)'$  i.e.

1.  $\gamma$  is an antilinear bijection
2.  $\gamma \circ \gamma = i$  where  $i$  is identity map
3.  $\gamma(T^*) = \gamma(A)^*$  for all  $T \in \sigma_{\Phi\#}(\mathfrak{M}_2)'$
4.  $\gamma(\{T, S\}) = \{\gamma(T), \gamma(S)\}$  for all  $T, S \in \sigma_{\Phi\#}(\mathfrak{M}_2)'$

we can define a conjugation  $\hat{J}$  on Hilbert space  $\mathcal{L}_{\Phi\#}$  [12] as  $\hat{J}T\Omega_{\Phi\#} = \gamma(T)\Omega_{\Phi\#}$  for all  $T \in \sigma_{\Phi\#}(\mathfrak{M}_2)'$ .

We remark that for the relationship (10), this Jordan map must necessarily satisfy the following property:

$$\gamma(\tau_{\Phi\#}(y))\Omega_{\Phi\#} = V_{\Phi\#}J_1y\Omega_1$$

for all  $y \in \mathfrak{M}'_1$ .

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